

Electronic Notes in Theoretical Computer Science 74 (2003)
 URL: <http://www.elsevier.nl/locate/entcs/volume74.html> 14 pages

The complexity space of a valued linearly ordered set

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Abstract

By a valued linearly ordered set (a \mathcal{VLOS} for short), we mean a pair (X, φ) such that X is a linearly ordered set and φ is a strictly increasing (= positive monotone) nonnegative real valued function. Clearly, any \mathcal{VLOS} is a valuation space.

Each $\mathcal{VLOS} (X, \varphi)$ generates a linear weightable quasi-metric d_φ on X whose conjugate is order preserving. We show that the Smyth completion of (X, d_φ) also admits the structure of a \mathcal{VLOS} .

On the other hand, M. Schellekens introduced in 1995, the theory of complexity spaces to develop a topological foundation for the complexity analysis of programs. Here, we introduce the so-called complexity space of a $\mathcal{VLOS} (X, \varphi)$ and discuss some of its properties. In particular, we show that it is weightable and preserves Smyth completeness of (X, d_φ) . We apply this complexity approach to the measurement of real numbers and discuss some advantages of our methods with respect to those that use the classical Baire metric.

Mathematics Subject Classification (2000): 54E35, 54F05, 68Q25.

Keywords: linearly ordered set, positive monotone function, valuation, weightable quasi-metric, Smyth completion, complexity space, alphabet, Baire metric.

1 Introduction and preliminaries

Throughout this paper we shall denote by \mathbb{R}^+ , ω and \mathbb{N} the set of nonnegative real numbers, the set of nonnegative integer numbers and the set of positive integer numbers, respectively.

¹ The authors acknowledge the support of the Spanish Ministry of Science and Technology, Plan Nacional I+D+I, and FEDER, under grant BFM2003-02302

Let us recall that a linear order on a nonempty set X is a (partial) order \preceq on X such that $x \preceq y$ or $y \preceq x$ for all $x, y \in X$. A linearly (ordered) set is a pair (X, \preceq) such that X is a nonempty set and \preceq is a (linear) order on X .

Let (X, \preceq) and (Y, \sqsubseteq) be two ordered sets. A mapping $f : X \rightarrow Y$ is said to be monotone if $f(x) \sqsubseteq f(y)$ whenever $x \preceq y$, and it is called positive monotone if $f(x) \sqsubset f(y)$ whenever $x \prec y$. In case that $(Y, \sqsubseteq) = (\mathbb{R}^+, \leq)$, with \leq the usual order on \mathbb{R}^+ , we will say that f is a (positive) monotone function on (X, \preceq) .

Note that if (X, \preceq) is a linearly ordered set, then f is positive monotone if and only if it is one-to-one.

Our main references for quasi-pseudo-metric spaces are [3] and [5].

Let us recall that a quasi-pseudo-metric on a set X is a nonnegative real-valued function d on $X \times X$ such that for all $x, y, z \in X$: (i) $d(x, x) = 0$, and (ii) $d(x, z) \leq d(x, y) + d(y, z)$.

In our context by a quasi-metric we mean a quasi-pseudo-metric d on X such that $d(x, y) = d(y, x) = 0$ if and only if $x = y$.

The restriction of a quasi-(pseudo-)metric d on X to any subset of X , will be also denoted by d if no confusion arises.

A quasi-(pseudo-)metric space is a pair (X, d) such that X is a (nonempty) set and d is a quasi-(pseudo-)metric on X .

As usual the associated order \leq_d of a quasi-metric space (X, d) is defined by $x \leq_d y \Leftrightarrow d(x, y) = 0$.

Each quasi-pseudo-metric d on a set X induces a topology $\mathcal{T}(d)$ on X which has as a base the family of open d -balls $\{B_d(x, r) : x \in X, r > 0\}$, where $B_d(x, r) = \{y \in X : d(x, y) < r\}$ for all $x \in X$ and $r > 0$.

We say that a quasi-metric d on an ordered set (X, \preceq) is order preserving if $x \leq_d y$ whenever $x \preceq y$.

If d is a quasi-(pseudo-)metric on X , then the function d^{-1} defined on $X \times X$ by $d^{-1}(x, y) = d(y, x)$, is also a quasi-(pseudo-)metric on X called the conjugate of d , and the function d^s defined on $X \times X$ by $d^s(x, y) = d(x, y) \vee d^{-1}(x, y)$ is a (pseudo-)metric on X .

A quasi-metric d on a set X is said to be bicomplete if d^s is a complete metric on X . In this case we say that (X, d) is a bicomplete quasi-metric space.

A simple but useful example of a (bicomplete) quasi-metric space consists of the pair (\mathbb{R}, u) , where u is the so-called upper quasi-metric on \mathbb{R} , which is defined by $u(x, y) = (y - x) \vee 0$, for all $x, y \in \mathbb{R}$. Observe that u^s is the Euclidean metric on \mathbb{R} .

2 Generating quasi-metrics from positive monotone functions

In this section we present a general method for generating linear weightable quasi-metrics from monotone functions on linearly ordered sets, which will

be useful later on. In addition, the conjugate quasi-metric is order preserving. The method is essentially an adaptation to our context of the well-known methods to generate metrics and quasi-metrics from valuations on lattices and from semivaluations on semilattices, respectively [1], [15].

Proposition 1. *Let φ be a monotone function on a linearly ordered set (X, \preceq) . Then the real-valued function d_φ defined on $X \times X$ by*

$$\begin{aligned} d_\varphi(x, y) &= \varphi(y) - \varphi(x) \quad \text{if } x \preceq y, \\ d_\varphi(x, y) &= 0 \quad \text{if } y \preceq x \end{aligned}$$

is a quasi-pseudo-metric on X such that $(d_\varphi)^{-1}$ is order preserving.

Proof. It is obvious that $d_\varphi(x, x) = 0$ for all $x \in X$.

Next we show that for all $x, y, z \in X$, $d_\varphi(x, z) \leq d_\varphi(x, y) + d_\varphi(y, z)$.

Indeed, if $z \preceq x$, then $d_\varphi(x, z) = 0$, so we only consider the case that $x \preceq z$. In such a case, if $z \preceq y$, then $x \preceq y$, and hence

$$d_\varphi(x, z) = \varphi(z) - \varphi(x) \leq \varphi(y) - \varphi(x) = d_\varphi(x, y).$$

If $y \preceq z$ and $x \preceq y$ we clearly obtain $d_\varphi(x, z) = d_\varphi(x, y) + d_\varphi(y, z)$. It remains to consider the case that $y \preceq z$ and $y \not\preceq x$; but then we have that $d_\varphi(x, z) = \varphi(z) - \varphi(x) \leq \varphi(z) - \varphi(y) = d_\varphi(y, z)$.

Finally, observe that if $x, y \in X$ satisfy $x \preceq y$, then, $(d_\varphi)^{-1}(x, y) = 0$. We conclude that $(d_\varphi)^{-1}$ is order preserving on X . ■

Proposition 2. *Let φ be a monotone function on a linearly ordered set (X, \preceq) . Then d_φ is a quasi-metric on X if and only if φ is positive monotone.*

Proof. We first suppose that d_φ is a quasi-metric. Let $x, y \in X$, and assume without loss of generality that $x \prec y$. If $\varphi(x) = \varphi(y)$, then $d_\varphi(x, y) = d_\varphi(y, x) = 0$, so $x = y$, a contradiction. Therefore $\varphi(x) < \varphi(y)$.

Conversely, if $d_\varphi(x, y) = d_\varphi(y, x) = 0$, then $\varphi(x) = \varphi(y)$. Since, by assumption, φ is positive monotone, $x = y$. ■

Note that if φ is a positive monotone function on a linearly ordered set (X, \preceq) , then \preceq is exactly the order $\leq_{(d_\varphi)^{-1}}$.

Example 1. On (\mathbb{R}^+, \leq) consider the identity function id . Clearly, the quasi-metric d_{id} is exactly the upper quasi-metric u on \mathbb{R}^+ defined in Section 1, i.e. $d_{id}(x, y) = (y - x) \vee 0$.

Example 2. Let $a \in \mathbb{R}^+ \setminus \{0\}$. Define $\varphi_a : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by $\varphi_a(x) = ax$. It is easy to see that φ_a is a positive monotone function which induces the quasi-metric d_{φ_a} defined by $d_{\varphi_a}(x, y) = a(y - x)$ if $x \leq y$ and $d_{\varphi_a}(x, y) = 0$ if $x > y$.

Weightable quasi-metric spaces were introduced by S.G. Matthews [6] as a

part of the study of denotational semantics of dataflow networks. The domain interval, the domain of words and the (dual) complexity space are interesting examples of weightable quasi-metric spaces which appear in several fields of Theoretical Computer Science (see, for instance, [6], [13], [9], [15]).

Let us recall that a quasi-metric space (X, d) is said to be weightable if there exists a function $w : X \rightarrow \mathbb{R}^+$ such that for all $x, y \in X$, $d(x, y) + w(x) = d(y, x) + w(y)$. The function w is said to be a weighting function for (X, d) and the quasi-metric d is weightable by the function w .

Observe that (\mathbb{R}^+, u) is weightable with weighting function the identity function on \mathbb{R}^+ (compare Example 1 above).

Proposition 3. *Let φ be a positive monotone function on a linearly ordered set (X, \preceq) . Then d_φ is a weightable quasi-metric on X with weighting function φ .*

Proof. Let $x, y \in X$, and assume that $x \preceq y$. Then $d_\varphi(x, y) + \varphi(x) = \varphi(y) - \varphi(x) + \varphi(x)$. On the other hand $d_\varphi(y, x) + \varphi(y) = \varphi(y)$. Thus $d_\varphi(x, y) + \varphi(x) = d_\varphi(y, x) + \varphi(y)$ for all $x, y \in X$. ■

In the light of Propositions 2 and 3 we propose the following notion.

Definition 1. A valued linearly ordered set (\mathcal{VLOS} for short) is a pair (X, φ) such that X is a linearly ordered set and φ is a positive monotone function on X .

According to [14], a quasi-metric d on a set X is said to be linear if the associated order \leq_d is linear. The construction given in Proposition 1 immediately shows that if (X, φ) is a \mathcal{VLOS} , then the quasi-metric d_φ is linear.

The last result of this section clarifies the relevance of the class \mathcal{VLOS} . Furthermore, it will permit us to simplify some proofs in Section 3.

Let (X, d) and (Y, e) be two quasi-metric spaces. A mapping $f : (X, d) \rightarrow (Y, e)$ is an isometry (from (X, d) into (Y, e)) provided that $e(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. It is well known that each isometry from (X, d) into (Y, e) is a one-to-one mapping. If there is an isometry from (X, d) onto (Y, e) we say that (X, d) and (Y, e) are isometric.

Proposition 4. *Let (X, φ) be a \mathcal{VLOS} . Then φ is an isometry from the quasi-metric space (X, d_φ) into the quasi-metric space (\mathbb{R}^+, u) .*

Proof. Let $x, y \in X$. If $x \preceq y$ we have $\varphi(x) \leq \varphi(y)$; otherwise we have $\varphi(y) \leq \varphi(x)$. So, in any case, we obtain

$$u(\varphi(x), \varphi(y)) = (\varphi(y) - \varphi(x)) \vee 0 = d_\varphi(x, y).$$

Thus φ is an isometry from (X, d_φ) into (\mathbb{R}^+, u) . ■

3 The Smyth completion in class \mathcal{VLOS}

Smyth completion constitutes an interesting topic from a domain theoretic point of view. In fact, Smyth presented in [16] and [17] a topological framework for denotational semantics based on the theory of Smyth complete (and totally bounded) quasi-uniform and quasi-metric spaces.

Since every weightable quasi-metric space is Smyth completable [4], it follows that for each $\mathcal{VLOS} (X, \varphi)$, the quasi-metric space (X, d_φ) is Smyth completable.

Let us recall that a quasi-metric space (X, d) is Smyth completable if and only if every left K -Cauchy sequence in (X, d) is a Cauchy sequence in (X, d^s) [15], where a sequence $(x_n)_{n \in \mathbb{N}}$ in (X, d) is left K -Cauchy [8] provided that for each $\varepsilon > 0$ there is $k \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ whenever $k \leq n \leq m$.

A quasi-metric space (X, d) is Smyth complete if and only if every left K -Cauchy sequence in (X, d) has a limit point in (X, d^s) [15].

It immediately follows from the preceding results that a quasi-metric space is Smyth complete if and only if it is bicomplete and Smyth completable. Hence, each weightable bicomplete quasi-metric space is Smyth complete.

In this section we shall prove that the bicompletion of any \mathcal{VLOS} is a (Smyth-complete) \mathcal{VLOS} .

Let us recall that a quasi-metric space (Y, q) is said to be a bicompletion of the quasi-metric space (X, d) if (Y, q) is a bicomplete quasi-metric space such that (X, d) is isometric to a dense subspace of the metric space (Y, q^s) . It is well known that each quasi-metric space (X, d) has an (up to isometry) unique bicompletion (\tilde{X}, \tilde{d}) (see [2], [12]).

The bicompletion (\tilde{X}, \tilde{d}) is constructed as follows.

Denote by \hat{X} the set of all Cauchy sequences in the metric space (X, d^s) . For each pair $\{x_n\}_{n \in \mathbb{N}}, \{y_n\}_{n \in \mathbb{N}}$ of elements of \hat{X} put $p(\{x_n\}_{n \in \mathbb{N}}, \{y_n\}_{n \in \mathbb{N}}) = \lim_{n \rightarrow \infty} d(x_n, y_n)$. Then p is a bicomplete quasi-pseudo-metric on \hat{X} . Now let:

$$\mathcal{R} = \{(\{x_n\}_{n \in \mathbb{N}}, \{y_n\}_{n \in \mathbb{N}}) \in \hat{X} \times \hat{X} : p^s(\{x_n\}_{n \in \mathbb{N}}, \{y_n\}_{n \in \mathbb{N}}) = 0\}.$$

Clearly \mathcal{R} is an equivalence relation on \hat{X} . Denote by \tilde{X} the quotient \hat{X}/\mathcal{R} . For each pair $[\{x_n\}_{n \in \mathbb{N}}], [\{y_n\}_{n \in \mathbb{N}}]$ in \tilde{X} define

$$\tilde{d}([\{x_n\}_{n \in \mathbb{N}}], [\{y_n\}_{n \in \mathbb{N}}]) = p(\{x_n\}_{n \in \mathbb{N}}, \{y_n\}_{n \in \mathbb{N}}).$$

In [2] and [12] it was independently proved that (\tilde{X}, \tilde{d}) is a bicomplete quasi-metric space such that (X, d) is isometric to a dense subspace of the metric space $(\tilde{X}, (\tilde{d})^s)$. Therefore (\tilde{X}, \tilde{d}) is the bicompletion of (X, d) . Furthermore $(\tilde{d})^s = \tilde{d}^s$ on \tilde{X} , and the bicompletion coincides with the standard completion when (X, d) is a metric space.

Definition 2. A $\mathcal{VLOS} (X, \varphi)$ is called bicomplete if d_φ is a bicomplete quasi-metric on X .

By a subspace of a $\mathcal{VLOS} (X, \varphi)$, we mean a $\mathcal{VLOS} (Y, \varphi|_Y)$ such that Y is a subset of X and $\varphi|_Y$ is the restriction of φ to Y .

Definition 3. An isometry from a $\mathcal{VLOS} (X, \varphi)$ to a $\mathcal{VLOS} (Y, \psi)$ is a monotone mapping $h : X \rightarrow Y$ such that $\psi(h(x)) = \varphi(x)$ for all $x \in X$.

Remark. Note that if $h : (X, \varphi) \rightarrow (Y, \psi)$ is an isometry, then h is one-to-one. Thus h and h^{-1} are positive monotone mappings.

Definition 4. Two $\mathcal{VLOS} (X, \varphi)$ and (Y, ψ) are said to be isometric if there is a isometry h from X onto Y .

Proposition 5. Let h be an isometry from a $\mathcal{VLOS} (X, \varphi)$ onto a $\mathcal{VLOS} (Y, \psi)$. Then the quasi-metric spaces (X, d_φ) and (Y, d_ψ) are isometric by h .

Proof. Let $x, y \in X$. If $h(x) \preceq h(y)$, then $d_\psi(h(x), h(y)) = \psi(h(y)) - \psi(h(x)) = \varphi(y) - \varphi(x)$. Since h^{-1} is monotone, $h^{-1}(h(x)) \preceq h^{-1}(h(y))$. Hence $x \preceq y$, so $d_\varphi(x, y) = \varphi(y) - \varphi(x)$. Therefore $d_\psi(h(x), h(y)) = d_\varphi(x, y)$. If $h(y) \preceq h(x)$ then $y \preceq x$ and $d_\psi(h(x), h(y)) = 0 = d_\varphi(x, y)$. ■

Definition 5. Let (X, φ) be a \mathcal{VLOS} . We say that a bicomplete $\mathcal{VLOS} (Y, \psi)$ is a bicompletion of (X, φ) if (X, φ) is isometric to a subspace of (Y, ψ) that is dense in the metric space $(Y, (d_\psi)^s)$.

We shall prove that each \mathcal{VLOS} has an (up to isometry) unique bicompletion.

Let (X, φ) be a \mathcal{VLOS} . By Proposition 3, the quasi-metric space (X, d_φ) is weightable with weighting function φ .

Now let $(\tilde{X}, \tilde{d}_\varphi)$ be the bicompletion of (X, d_φ) . By Theorem 1 of [7], $(\tilde{X}, \tilde{d}_\varphi)$ is weightable with weighting function $\tilde{\varphi}$ given by $\tilde{\varphi}([x]) = \lim_{n \rightarrow \infty} \varphi(x_n)$ for all $x := (x_n)_{n \in \mathbb{N}} \in \hat{X}$.

Next we define a binary relation \sqsubseteq on \tilde{X} as follows:

For each pair $x := \{x_n\}_{n \in \mathbb{N}}$ and $y := \{y_n\}_{n \in \mathbb{N}}$ of elements of \hat{X} let

$$[x] \sqsubseteq [y] \Leftrightarrow \tilde{\varphi}([x]) \leq \tilde{\varphi}([y]).$$

Clearly (\tilde{X}, \sqsubseteq) is a linearly ordered set, and we immediately obtain the following result.

Lemma 1. Let (X, φ) be a \mathcal{VLOS} . Then $(\tilde{X}, \tilde{\varphi})$ is a \mathcal{VLOS} .

By Propositions 2 and 3 and the preceding lemma, $d_{\tilde{\varphi}}$ is a weightable quasi-metric on \tilde{X} with weighting function $\tilde{\varphi}$.

Lemma 2. $\widetilde{d}_\varphi = d_{\widetilde{\varphi}}$ on \widetilde{X} .

Proof. Let $x := \{x_n\}_{n \in \mathbb{N}}$ and $y := \{y_n\}_{n \in \mathbb{N}}$ be two elements of \widehat{X} . Suppose without loss of generality that $[x] \sqsubset [y]$. Then $\widetilde{\varphi}([x]) < \widetilde{\varphi}([y])$, and hence $\varphi(x_n) < \varphi(y_n)$ eventually. Therefore

$$\begin{aligned} \widetilde{d}_\varphi([x], [y]) &= \lim_{n \rightarrow \infty} d_\varphi(x_n, y_n) = \lim_{n \rightarrow \infty} (\varphi(y_n) - \varphi(x_n)) \\ &= \lim_{n \rightarrow \infty} \varphi(y_n) - \lim_{n \rightarrow \infty} \varphi(x_n) = \widetilde{\varphi}([y]) - \widetilde{\varphi}([x]) \\ &= d_{\widetilde{\varphi}}([x], [y]). \end{aligned}$$

We conclude that $\widetilde{d}_\varphi = d_{\widetilde{\varphi}}$ on \widetilde{X} . ■

Corollary. $(\widetilde{X}, \widetilde{\varphi})$ is a bicomplete \mathcal{VLOS} .

Lemma 3. $(\widetilde{X}, \widetilde{\varphi})$ is a bicompletion of (X, φ) .

Proof. For each $x \in X$ denote by \widehat{x} the constant sequence x, x, \dots, x, \dots . Since, by Lemma 2, $(\widetilde{X}, d_{\widetilde{\varphi}})$ is the (quasi-metric) bicompletion of the quasi-metric space (X, d_φ) , $i(X)$ is dense in $(\widetilde{X}, (d_{\widetilde{\varphi}})^s)$ where i denotes the one-to-one mapping from X to \widetilde{X} given by $i(x) = [\widehat{x}]$ for all $x \in X$. Note that $[\widehat{x}]$ consists of all sequences in X which converge to x in the metric space $(X, (d_\varphi)^s)$. Clearly i is an monotone function, so $i(X)$ is a linearly ordered subset of $(\widetilde{X}, \sqsubseteq)$. Finally, since $\widetilde{\varphi}(i(x)) = \widetilde{\varphi}([\widehat{x}]) = \varphi(x)$ for all $x \in X$, we deduce that (X, φ) and $(i(X), \widetilde{\varphi}|_{i(X)})$ are isometric \mathcal{VLOS} . The proof is complete. ■

Lemma 4. Let (X, φ) be a \mathcal{VLOS} . Then any bicompletion of (X, φ) is isometric to $(\widetilde{X}, \widetilde{\varphi})$.

Proof. Let (Y, ψ) be a bicompletion of (X, φ) . By Proposition 5, (Y, d_ψ) is a (quasi-metric) bicompletion of (X, d_φ) . Since the bicompletion of a quasi-metric space is unique up to isometry, there is an isometry h from (Y, d_ψ) onto $(\widetilde{X}, d_{\widetilde{\varphi}})$. We want to show that h is an isometry from (Y, ψ) onto $(\widetilde{X}, \widetilde{\varphi})$.

Indeed, let $x, y \in Y$ such that $x \prec y$. Since ψ is positive monotone, we have

$$0 < \psi(y) - \psi(x) = d_\psi(x, y) = d_{\widetilde{\varphi}}(h(x), h(y)),$$

Hence $h(x) \sqsubset h(y)$. Thus h is (positive) monotone.

On the other hand, it follows from Remark 2 of [7] that $\varphi(h)$ is a weighting function for the quasi-metric d_ψ on Y . Then, there is a constant function c such that $\varphi(h) = \psi + c$ (see, for instance, Proposition 3.2 of [10]). Since for each $x \in X$, we have $\varphi(h(x)) = \psi(x)$, it follows that $c = 0$, and consequently $\varphi(h(y)) = \psi(y)$ for all $y \in Y$.

We conclude that (Y, ψ) and $(\widetilde{X}, \widetilde{\varphi})$ are isometric \mathcal{VLOS} . ■

From the above lemmas and the fact, cited above, that each weightable bicomplete quasi-metric space is Smyth complete, we deduce the following.

Theorem 1. *Each $\mathcal{VLOS} (X, \varphi)$ has a bicompletion $(\tilde{X}, \tilde{\varphi})$ which is unique up to isometry. Furthermore $(\tilde{X}, d_{\tilde{\varphi}})$ is Smyth complete.*

4 The complexity space of a \mathcal{VLOS}

Let (X, φ) be a \mathcal{VLOS} . Set

$$C_{X,\varphi}^* = \{f \in X^\omega : \sum_{n=0}^{\infty} 2^{-n} \varphi(f(n)) < +\infty\}$$

and

$$d_{C_{X,\varphi}^*}(f, g) = \sum_{n=0}^{\infty} 2^{-n} d_{\varphi}(f(n), g(n))$$

for all $f, g \in C_{X,\varphi}^*$. Note that $C_{X,\varphi}^* \neq \emptyset$, because constant mappings are in $C_{X,\varphi}^*$.

Note that by Proposition 4, we have

$$d_{C_{X,\varphi}^*}(f, g) = \sum_{n=0}^{\infty} 2^{-n} [(\varphi(g(n)) - \varphi(f(n))) \vee 0].$$

By $f \preceq g$ we mean that $f(n) \preceq g(n)$ for all $n \in \omega$. Then $(C_{X,\varphi}^*, \preceq)$ is clearly an ordered set.

It is easy to see that $d_{C_{X,\varphi}^*}$ is a quasi-metric on $C_{X,\varphi}^*$ whose conjugate quasi-metric is order preserving, and analogous to [11], the quasi-metric space $(C_{X,\varphi}^*, d_{C_{X,\varphi}^*})$ will be called the complexity space of (X, φ) , and $d_{C_{X,\varphi}^*}$ the complexity quasi-metric of (X, φ) . In particular, if $X = \mathbb{R}^+$ and φ is the identity function id on \mathbb{R}^+ , then the complexity space of the $\mathcal{VLOS} (\mathbb{R}^+, id)$ is the so-called dual complexity space (see [9]), which consists of the pair $(\mathcal{C}^*, d_{\mathcal{C}^*})$, where

$$\mathcal{C}^* = \{f \in (\mathbb{R}^+)^{\omega} : \sum_{n=0}^{\infty} 2^{-n} f(n) < +\infty\},$$

and $d_{\mathcal{C}^*}$ is the quasi-metric on \mathcal{C}^* given by

$$d_{\mathcal{C}^*}(f, g) = \sum_{n=0}^{\infty} 2^{-n} [(g(n) - f(n)) \vee 0].$$

Proposition 6. *Let (X, φ) be a \mathcal{VLOS} . Then the mapping*

$$\Psi : (C_{X,\varphi}^*, d_{C_{X,\varphi}^*}) \rightarrow (\mathcal{C}^*, d_{\mathcal{C}^*})$$

given by the rule

$$\Psi(f)(n) = \varphi(f(n))$$

is a positive monotone isometry.

Proof. First note that Ψ is well-defined because for each $f \in C_{X,\varphi}^*$ one has $\sum_{n=0}^{\infty} 2^{-n}\varphi(f(n)) < +\infty$, and thus $\Psi(f) \in \mathcal{C}^*$.

Now let $f, g \in C_{X,\varphi}^*$. Then

$$\begin{aligned} d_{\mathcal{C}^*}(\Psi(f), \Psi(g)) &= \sum_{n=0}^{\infty} 2^{-n}[(\Psi(g)(n) - \Psi(f)(n)) \vee 0] \\ &= \sum_{n=0}^{\infty} 2^{-n}[(\varphi(g(n)) - \varphi(f(n))) \vee 0] = d_{C_{X,\varphi}^*}(f, g). \end{aligned}$$

Hence Ψ is an isometry from $(C_{X,\varphi}^*, d_{C_{X,\varphi}^*})$ into $(\mathcal{C}^*, d_{\mathcal{C}^*})$.

Finally, let $f, g \in C_{X,\varphi}^*$ such that $f \prec g$. Since φ is positive monotone it follows that $\Psi(f) < \Psi(g)$. We conclude that Ψ is positive monotone. ■

It is well known [9] that the dual complexity space $(\mathcal{C}^*, d_{\mathcal{C}^*})$ is a weightable quasi-metric space with weighting function W given by $W(f) = \sum_{n=0}^{\infty} 2^{-n}f(n)$. Combining this result with Proposition 6 we deduce the following.

Proposition 7. *The complexity space $(C_{X,\varphi}^*, d_{C_{X,\varphi}^*})$ is weightable with weighting function W_{φ} defined on $C_{X,\varphi}^*$ by $W_{\varphi}(f) = \sum_{n=0}^{\infty} 2^{-n}\varphi(f(n))$.*

It was proved in [9] that the dual complexity space is Smyth complete. In our next theorem we extend this result to any complexity space $(C_{X,\varphi}^*, d_{C_{X,\varphi}^*})$.

Theorem 2. *Let (X, φ) be a \mathcal{VLOS} . Then the following statements are equivalent.*

- (1) $(C_{X,\varphi}^*, d_{C_{X,\varphi}^*})$ is bicomplete.
- (2) $(C_{X,\varphi}^*, d_{C_{X,\varphi}^*})$ is Smyth complete.
- (3) (X, d_{φ}) is Smyth complete
- (4) (X, d_{φ}) is bicomplete.

Proof. (1) \Rightarrow (2). Since the complexity space $(C_{X,\varphi}^*, d_{C_{X,\varphi}^*})$ is a weightable bicomplete quasi-metric space, we deduce that it is Smyth complete.

(2) \Rightarrow (3). Let $\{x_k\}_{k \in \mathbb{N}}$ be a left K -Cauchy sequence in (X, d_{φ}) . Consider the sequence $\{f_k\}_{k \in \mathbb{N}}$ in $(C_{X,\varphi}^*, d_{C_{X,\varphi}^*})$ where each $f_k : \omega \rightarrow X$ is the constant mapping defined by $f_k(n) = x_k$. Next we show that $\{f_k\}_{k \in \mathbb{N}}$ is a left K -Cauchy sequence in $(C_{X,\varphi}^*, d_{C_{X,\varphi}^*})$. Indeed, for each $\varepsilon > 0$ there exists $k_{\varepsilon} \in \mathbb{N}$ such that $d_{\varphi}(x_k, x_j) < \varepsilon/2$ whenever $j \geq k \geq k_{\varepsilon}$. Thus

$$d_{C_{X,\varphi}^*}(f_k, f_j) = \sum_{n=0}^{\infty} 2^{-n}d_{\varphi}(f_k(n), f_j(n)) =$$

$$\sum_{n=0}^{\infty} 2^{-n} d_{\varphi}(x_k, x_j) < \frac{\varepsilon}{2} \sum_{n=0}^{\infty} 2^{-n} = \varepsilon$$

whenever $j \geq k \geq k_{\varepsilon}$. Since $(C_{X,\varphi}^*, d_{C_{X,\varphi}^*})$ is Smyth complete, there is $f \in C_{X,\varphi}^*$ such that $\lim_{k \rightarrow \infty} (d_{C_{X,\varphi}^*})^s(f, f_k) = 0$. Put $y = f(0)$. Since

$$\begin{aligned} (d_{\varphi})^s(y, x_k) &\leq \sum_{n=0}^{\infty} 2^{-n} d_{\varphi}(f(n), f_k(n)) + \sum_{n=0}^{\infty} 2^{-n} d_{\varphi}(f_k(n), f(n)) \\ &= d_{C_{X,\varphi}^*}(f, f_k) + d_{C_{X,\varphi}^*}(f_k, f), \end{aligned}$$

it immediately follows that $\{x_k\}_{k \in \mathbb{N}}$ converges to y in $(X, (d_{\varphi})^s)$. Therefore (X, d_{φ}) is Smyth complete.

(3) \Rightarrow (4). Obvious.

(4) \Rightarrow (1). Since by Proposition 4, (X, d_{φ}) is isometric to the subspace $(\varphi(X), u)$ of (\mathbb{R}^+, u) , it follows from our assumption that $(\varphi(X), u)$ is bicomplete (note that actually it is Smyth complete).

Fix $y \in \varphi(X)$. According to the terminology of [11], let

$$\mathcal{B}_y = \{f \in (\varphi(X))^{\omega} : \sum_{n=0}^{\infty} 2^{-n} |y - f(n)| < +\infty\},$$

and let $u_{\mathcal{B}_y}$ be the quasi-metric on \mathcal{B}_y given by $u_{\mathcal{B}_y}(f, g) = \sum_{n=0}^{\infty} 2^{-n} u(f(n), g(n))$. By Theorem 1 of [11], $(\mathcal{B}_y, u_{\mathcal{B}_y})$ is a bicomplete quasi-metric space.

Next we observe that the $\Psi(C_{X,\varphi}^*) = \mathcal{B}_y$, where Ψ is the isometry defined in Proposition 6.

Indeed, let $f \in C_{X,\varphi}^*$. Then

$$\sum_{n=0}^{\infty} 2^{-n} |y - \varphi(f(n))| \leq \sum_{n=0}^{\infty} 2^{-n} y + \sum_{n=0}^{\infty} 2^{-n} \varphi(f(n)) < +\infty,$$

so $\Psi(f) \in \mathcal{B}_y$. Now let $f \in \mathcal{B}_y$. For each $n \in \omega$ there is $x_n \in X$ such that $f(n) = \varphi(x_n)$. Define $h \in X^{\omega}$ by $h(n) = x_n$ for all $n \in \omega$. Since $f \in \mathcal{B}_y$ it immediately follows that $\sum_{n=0}^{\infty} 2^{-n} f(n) < +\infty$, and thus $\sum_{n=0}^{\infty} 2^{-n} \varphi(h(n)) < +\infty$. Therefore $h \in C_{X,\varphi}^*$ and $\Psi(h) = f$.

We conclude that $\Psi(C_{X,\varphi}^*) = \mathcal{B}_y$. Then $(C_{X,\varphi}^*, d_{C_{X,\varphi}^*})$ is isometric to $(\mathcal{B}_y, u_{\mathcal{B}_y})$ by Proposition 6, and consequently $(C_{X,\varphi}^*, d_{C_{X,\varphi}^*})$ is bicomplete. ■

It is well known ([9]) that if $g \in \mathcal{C}^*$, then $(\mathcal{C}_g^*, (d_{\mathcal{C}^*})^s)$ is a compact metric space, where $\mathcal{C}_g^* = \{f \in \mathcal{C}^* : f \leq g\}$. From this result and Proposition 6 we deduce the following.

Theorem 3. Let (X, φ) be a \mathcal{VLOS} and let $g \in C_{X,\varphi}^*$, then $(C_g^*, (d_{C_{X,\varphi}^*})^s)$ is a compact metric space, where $C_g^* = \{f \in C_{X,\varphi}^* : f \leq g\}$.

We conclude the paper with an application of the complexity quasi-metrics to the measurement of distances between infinite words over the decimal al-

phabet, and analyze some advantages of our methods with respect to those that use the classical Baire metric.

Let $\Sigma = \{0, 1, 2, \dots, 9\}$ and let $\varphi : \Sigma \rightarrow \mathbb{R}^+$ defined by $\varphi(x) = 2^{-(10-x)}$ for all $x \in \Sigma$. Then (Σ, φ) is a \mathcal{VLOS} , where Σ is equipped with the restriction of the usual order on \mathbb{R} .

Obviously $(\Sigma, (d_\varphi)^s)$ is a compact metric space, so in particular (Σ, d_φ) is Smyth complete.

Denote by Σ^ω the set of all infinite words over Σ . Each $w \in \Sigma^\omega$ will be expressed by $w_0w_1w_2\dots$, or by $(w_n)_{n \in \omega}$ if no confusion arises.

Clearly, we may assume that the complexity space of (Σ, φ) is the pair $(\Sigma^\omega, d_{\Sigma^\omega})$ where d_{Σ^ω} is the weightable quasi-metric on Σ^ω given by

$$d_{\Sigma^\omega}(v, w) = \sum_{n=0}^{\infty} 2^{-n} d_\varphi(v_n, w_n),$$

i.e.

$$d_{\Sigma^\omega}(v, w) = \sum_{n=0}^{\infty} 2^{-n} \left[\left(\frac{1}{2^{10-w_n}} - \frac{1}{2^{10-v_n}} \right) \vee 0 \right].$$

It follows from Theorem 2 that $(\Sigma^\omega, d_{\Sigma^\omega})$ is Smyth complete.

A typical and well-known metric on Σ^ω is the so-called Baire metric which is given by

$$D(v, w) = 2^{-\ell(v, w)} \quad \text{if } v \neq w, \quad \text{and} \quad D(w, w) = 0,$$

for all $v, w \in \Sigma^\omega$, where $\ell(v, w)$ is defined as the length of the nonempty common prefix of v and w if they exist, and $\ell(v, w) = 0$ otherwise.

The following result establishes a useful relation between metrics D and $(d_{\Sigma^\omega})^s$.

Proposition 8. *For each $v, w \in \Sigma^\omega$ we have*

$$2^{-11} D(v, w) \leq (d_{\Sigma^\omega})^s(v, w) \leq (1 - 2^{-9}) D(v, w).$$

Proof. Let $v, w \in \Sigma^\omega$. We assume that $v \neq w$. Then $D(v, w) = 2^{-\ell(v, w)}$. Put $\ell(v, w) = k$. Since

$$\begin{aligned} (d_{\Sigma^\omega})^s(v, w) &\leq \sum_{n=0}^{\infty} 2^{-n} \left| \frac{1}{2^{10-w_n}} - \frac{1}{2^{10-v_n}} \right| = d_{\Sigma^\omega}(v, w) + d_{\Sigma^\omega}(w, v) \\ &\leq 2(d_{\Sigma^\omega})^s(v, w), \end{aligned}$$

it follows

$$\begin{aligned} (d_{\Sigma^\omega})^s(v, w) &\leq \sum_{n=k}^{\infty} 2^{-n} \left| \frac{1}{2^{10-w_n}} - \frac{1}{2^{10-v_n}} \right| \leq \left(\frac{1}{2} - \frac{1}{2^{10}} \right) \sum_{n=k}^{\infty} 2^{-n} \\ &= \frac{2^9 - 1}{2^{10}} 2^{-(k-1)} = \frac{2^9 - 1}{2^9} D(v, w). \end{aligned}$$

and

$$\begin{aligned} 2(d_{\Sigma^\omega})^s(v, w) &\geq \sum_{n=k}^{\infty} 2^{-n} \left| \frac{1}{2^{10-w_n}} - \frac{1}{2^{10-v_n}} \right| \geq 2^{-k} \left(\frac{1}{2^9} - \frac{1}{2^{10}} \right) \\ &= 2^{-(k+10)} = 2^{-10} D(v, w). \end{aligned}$$

This completes the proof. ■

As a consequence we obtain the following well-known result.

Corollary. (Σ^ω, D) is complete.

The following technical result will be useful in the rest of the paper. We omit its easy proof.

Proposition 9. Let $u, v, w \in \Sigma^\omega$ be such that $u_n \leq v_n$ and $u_n \leq w_n$ for all $n \in \omega$. If there is $n_0 \in \omega$ such that $v_n \leq w_n$ for all $n \geq n_0$, then

$$d_{\Sigma^\omega}(u, w) - d_{\Sigma^\omega}(u, v) \geq 2^{-n_0} \left(\frac{1}{2^{10-w_{n_0}}} - \frac{1}{2^{10-v_{n_0}}} \right).$$

To discuss in our context some advantages of the complexity quasi-metric with respect to the Baire metric, we focus, without loss of generality, in computing distances on the interval $[0, 1]$.

For each real number in $]0, 1[$ admitting a rational decimal expansion, choose exactly this expansion. In addition, identify the real number 0 with 0.000..., and the real number 1 with 0.999...

Consider the unique decimal expansion of all numbers in the interval $[0, 1]$ that is obtained in this way, and denote by Ω the set of such expansions.

Let $\Sigma_0^\omega = \{w \in \Sigma^\omega : w_0 = 0\}$. Since each $x \in \Omega$ can be viewed as an element of Σ_0^ω , we assume without loss of generality that Ω is a subset of Σ^ω , in the sequel.

Observe that if $\{v_k\}_{k \in \mathbb{N}}$ and $\{w_k\}_{k \in \mathbb{N}}$ are sequences in Ω with $v_k \neq w_k$ for all $k \in \mathbb{N}$, and there is $w \in \Omega$ such that $D(w, v_k) \rightarrow 0$, $D(w, w_k) \rightarrow 0$ and $\ell(w, v_k) = \ell(w, w_k)$, then $D(w, v_k) = D(w, w_k)$ for all $k \in \mathbb{N}$. However, Proposition 9 shows that the quasi-metric d_{Σ^ω} , and thus the metric $(d_{\Sigma^\omega})^s$, is able to distinguish between the “distances” from w to v_k and from w to w_k , respectively, in many interesting cases.

We illustrate this fact with the following example.

Example 3. Denote simply by 0 the word 00000...

Consider the sequence $\{v_k\}_{k \in \mathbb{N}}$ in Ω given by

$$v_1 := 01000...$$

$$v_2 := 00100...$$

.....

$$v_k := \overbrace{000 \dots 0}^{k\text{-times}} 1000...$$

Clearly we obtain $D(0, v_k) = 2^{-k}$ for all $k \in \mathbb{N}$.

On the other hand

$$d_{\Sigma^\omega}(0, v_k) = 2^{-k}[\varphi(1) - \varphi(0)] = 2^{-k}\left[\frac{1}{2^{10-1}} - \frac{1}{2^{10}}\right] = 2^{-k}\frac{1}{2^{10}}$$

and hence

$$d_{\Sigma^\omega}(0, v_k) = 2^{-10}D(0, v_k).$$

for all $k \in \mathbb{N}$.

Now, we take the sequence $\{w_k\}_{k \in \mathbb{N}}$ in Ω given by

$$w_1 := 011000\dots$$

$$w_2 := 001100\dots$$

$$\dots\dots\dots$$

$$w_k := \overbrace{000\dots 0}^{k\text{-times}} 11000\dots$$

A straightforward calculation shows that $D(0, w_k) = 2^{-k}$ for all $k \in \mathbb{N}$.

Hence $D(0, v_k) = D(0, w_k)$ for all $k \in \mathbb{N}$, and, obviously, $D(w, v_k) \rightarrow 0$ and $D(w, w_k) \rightarrow 0$.

However, by Proposition 9,

$$d_{\Sigma^\omega}(0, w_k) - d_{\Sigma^\omega}(0, v_k) \geq 2^{-(k+1)}\frac{1}{2^{10}},$$

which provides a reasonable and desirable relation. Note that the inequality sign is, actually, an equality sign for this case.

Therefore, we exactly obtain

$$d_{\Sigma^\omega}(0, w_k) = d_{\Sigma^\omega}(0, v_k) + 2^{-(k+1)}\frac{1}{2^{10}} = 2^{-(k+1)}\frac{3}{2^{10}}$$

and, thus

$$d_{\Sigma^\omega}(0, w_k) = \frac{3}{2^{11}}D(0, w_k)$$

for all $k \in \mathbb{N}$.

Acknowledgement. The authors are very grateful to Professor Mícheál Mac an Airchinnigh for a careful review of the paper.

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